Efficiency in Economies with Jurisdictions and Local Public Projects*

Julian R A Manning†

August 9, 1998

Abstract

Traditionally in the literature on local public goods it is assumed that each local public good is a selection from a convex space. In this paper existence of equilibrium is shown for a class of finite models where local public goods are selections from abstract, possibly non-convex, commodity spaces. Consumers are free to migrate between regions. Equilibrium are supported by a system of personalized valuations. It is demonstrated that consumers not only must face a system of personalized valuations in equilibrium but must also, in general, face a different system of personalized valuations out of equilibrium. These equilibria are shown to lie in the core.

*Version 3.1. This is a revised version of Chapter 4 of my thesis [12] and a rewrite of the first paper derived from Chapter 4 of my thesis that had circulated under the title "Efficiency in Economies with Jurisdictions and Public Projects" [13]. The author wishes to thank Marcus Berliant, Dimitrios Diamantaras, Robert Gilles and Myrna Wooders for their comments on an earlier version of this paper. Responsibility for all errors is mine.

†Please send all correspondence to Julian R A Manning, Department of Business Economics, Norwegian School of Management, Elias Smiths vei 15, PO Box 580, N-1301 Sandvika, Norway, Phone: 47 67 57 07 96, Fax: 47 67 57 05 70, E-mail: Julian.Manning@bi.no

1 Introduction

In this paper we introduce a system of potentially non-linear prices, we call valuations, that support Pareto optimal allocations as equilibrium for a model with a finite number of consumers and a central government free to choose a public project from a set of public projects for each of a finite number of regions. Consumers are free to migrate between regions. Equilibrium are supported by a system of personalized valuations. It is demonstrated that consumers not only must face a system of personalized valuations in equilibrium but must also, in general, face a different system of personalized valuations out of equilibrium.[1]

Tiebout (1956) theorized that, with a sufficiently large number of jurisdictions, migration would lead to near efficient provision of local public goods. In modelling local public good economies, Tiebout imposed no restrictions on the nature of the local public goods except that consumption by any one consumer does not diminish the quantity of any local public good available for any other consumer and local public goods are not excludable within each region. Any consumer who resides in the region in which a local public good is provided may consume the public good. Since Tiebout, a number of papers have clarified when Tiebout equilibria exist and when Tiebout equilibria are Pareto optimal. However, it is traditional in the literature that each local public good may easily be quantified by picking a number on the real line and that each local public good is infinitely divisible. This is true of Bewley (1981), Ellickson (1973) and Wooders (1978, 1980, 1989). It is clear that not all public goods may be infinitely divisible or characterized by picking a number on the real line (for example, building a bridge over a river or the exploration of outer space).

In Mas-Colell (1980), Mas-Colell introduced the concept of a valuation function to allow Pareto optimal allocations of pure public goods to be implemented where the pure public goods are elements of an abstract (possibly non-convex) commodity space. An element of such a commodity space is called a public project. A valuation function assigns, for each element of the

[1]These valuations have more recently been called "conjectural" valuations. Consumers have to know that the valuations they face may be different out of equilibrium so that the equilibrium allocation is the "best" affordable allocation under the equilibrium allocation of consumers amongst regions and the "out of equilibrium" allocation of consumers amongst regions.

commodity space, an individual value for each consumer. This valuation is interpreted as the amount each consumer must pay to consume that public project. In this paper it is shown that Pareto optimal allocations of local public projects may be supported with a valuation system. The form of valuation system required depends on how each consumers preferences change as any consumer migrates. Further, in this paper it is shown that if the concept of valuation equilibrium is restricted in such a way that the valuation of each public project must be non-negative then the set of "cost share" valuation equilibrium lie in the core.

In Section 2 I describe the class of models used to demonstrate the Propositions to follow. An allocation consists of an assignment of a local public project to each region, private goods to each consumer and the assignment of consumers to regions. Each assignment of local public projects to regions is an element of an abstract (possibly non-convex) commodity space. Three candidate valuation systems are considered. Each valuation system is an analogue of a price system to be found in Manning (1993) and Wooders (1989) in models with infinitely divisible local public goods. Production in this paper and Mas-Colell (1980) may be viewed as performed by a central government. Unlike Diamantaras and Gilles (1993), all consumers may affect production opportunities. Production of the local public project may therefore be interpreted as potentially involving the services of all consumers in that jurisdiction.

The class of models with local public projects I introduce allows for any form of congestion in consumption and production.

In Section 3 three Second Welfare Theorems are proved for classes of models with complete personalized, personalized and non-personalized valuation systems. A complete personalized valuation system is a valuation system such that any consumer's valuation may change as he (she) or any other consumer migrates. In the first Second Welfare Theorem, non-anonymous crowding in consumption and production may occur. It is shown that any Pareto optimal allocation may be supported as a complete personalized valuation system. In the second Second Welfare Theorem, the crowding in consumption is restricted. Consumers may not enjoy higher utility associated with the same allocations as the assignment of consumers to regions changes. In addition, production opportunities do not expand as consumers move away from the Pareto optimal assignment of consumers to regions. As a consequence, it is shown that the valuation system may be constrained to the set of personalized valuation systems. A personalized valuation system is a valuation system such that any consumer's valuation of a local public project may differ from that of any other consumer but will not change as he (she) or any other consumer migrates. In the third Second Welfare Theorem consumer preferences are further restricted to be globally the same. Therefore, the valuation system may be constrained to the set of non-personalized valuation systems. A non-personalized valuation system is a valuation system such that every consumer's valuation of each local public project is the same.

Unlike models with infinitely divisible public goods and linear Lindahl prices, residence taxes are never needed. Valuations may be chosen such that residence taxes are implicit.

In section 4 continuity of the prices of private goods is examined. Extensions of the model in Section 2 are considered in Section 5. It is shown that the results of Section 3 and Section 4 are robust to the introduction of decentralized production, intermediate local public projects and the production of private goods. In addition, conditions on the primitives sufficient to ensure that the valuations faced by any consumer only depend on "regional" characteristics (the characteristics of other consumers in the same region) are given.

All proofs appear in an Appendix.

2 The Model

The model may briefly be described as follows: there are L private commodities, aggregate consumption of which is denoted by $x \in \Re^L$, there are I consumers and J regions. We use the convention $\mathcal{L} = \{1, \ldots, l, \ldots, L\}$ and similarly for $\mathcal{I}$ and $\mathcal{J}$.

Allocations of consumers among regions are written as partitions of $\mathcal{I}$, S, where $\#S = J$. The set of all possible partitions of the set of all consumers $\mathcal{I}$ is Z_J. In Sections 4 and 5 the constraint that there be J regions is relaxed so that the set of all possible partitions of consumers becomes Z. All the definitions to follow in this section extend from Z_J to Z in a natural fashion.

Associated with each region j is a local public project y_S^j. Often we will write the vector of local public projects $(y_S^j)_J$ as y_S. If the vector of local public projects does not change as the partition of consumers changes, say from S_1 to S_2, then we say $y_{S_1} = y_{S_2}$.

2.1 Production

The set of all possible vectors of local public projects is Y_J, where $Y_J = \Pi^{\star}_{Z_J} Y_S$ and where Y_S is the set of all possible vectors of local public projects available under the partition of consumers S.[2] It is possible that no local public project is provided in any region so $0 \in Y_S$ for all partitions S. The cost of any vector of local public projects is given by a function $C\colon Y_J \to \Re^L_+$. The cost function C is assumed throughout to be proper in Y_J.[3] This ensures that the set of feasible allocations is compact.

A *production program* is a $J+L$-tuple (y_S, x_S) where x_S is the vector of private goods left after production of local public projects y_S, both relative to the partition S.

2.2 Consumers

Consumers are free to reside or not to reside in any region.

Each consumer is only endowed with a non-negative vector of private commodities $w^i \in \Re^L_+ \setminus \{0\}$.

If any consumer resides in region j then that consumers consumption of local public projects is $(0, \ldots, y^j_S, \ldots, 0)$ or more simply $\underline{y}^j_S$. The consumption vector of consumer i, z^i_S, when residing in region j is $(\underline{y}^j_S, x^i_S)$, where x^i_S is vector of private commodities consumed and $\underline{y}^j_S$ is the consumption of local public projects by consumer i relative to the partition S. Aggregate consumption is therefore z_S where $z_S = (y_S, x_S)$ and $x_S = \sum x^i_S$.

Each consumer's utility is a function of his (her) consumption of public projects, money and the composition of the population of the region in which he (she) resides. If consumer i resides in region j then the presence of consumer i provides a service (disservice) to all residents in region j. Further,

[2] Suppose that $\{A_{S_1}, \ldots, A_{S_m}\}$ is a collection of sets, indexed by the partitions $Z = \{S_1, \ldots, S_k, \ldots, S_m\}$. We define the *star product* of this indexed collection of sets, denoted by $\Pi^{\star}_Z A_{S_k}$, to be the set of all m-tuples $(\emptyset, \ldots, x_{S_k}, \ldots, \emptyset)$ such that $x_{S_k} \in A_{S_k}$, for each $k = 1, ..., m$. Instead of writing elements of $\Pi^{\star}_Z A_{S_k}$ in m-tuple form they will be collapsed to 1-tuple form as follows: $(\emptyset, \ldots, x_{S_k}, \ldots, \emptyset)$ is written as x_{S_k} for all $k = 1, ..., m$.

[3] That C is *proper* in Y means that the preimage of any compact set in $\Re^L$ is a compact set in Y.

the presence of each resident of region j provides a service (disservice) to consumer i. Therefore, the commodity space of each consumer is taken to include the residence choice of all consumers. Preferences for each consumer i are represented by the complete preordering $\succ^i$over X^i_J where $X^i_J = \Pi^\star_{Z_J} X^i_S$ and $X^i_S \subset Y_S \times \Re^L$, for all partitions S in Z_J. We assume that $w^i \in X^i_S$, for all partitions S in Z_J.

In general each consumer pays a *complete personalized valuation* on Z_J for local public projects $V: \mathcal{I} \times Y_J \rightarrow \Re$. The function V can be represented as a vector $V = (V^i(y_S))$. It is assumed throughout that each V^i is an upper semi-continuous function $V^i: Y_J \rightarrow \Re$ for all consumers i.

Sometimes we can restrict the valuation each consumer pays to a *personalized valuation* for local public projects or a *non-personalized valuation* for local public projects charasterized by property (1) or properties (1) and (2), respectively.

Property 1 for every consumer i, $V^i(y_S) = V^i(y_{\hat{S}})$ for every (y_S, x^i_S) in X^i_S, $(y_{\hat{S}}, \hat{x}^i_{\hat{S}})$ in $X^i_{\hat{S}}$, $S, \hat{S}$ in Z_J, such that $y_S = y_{\hat{S}}$.

Property 2 for every pair of consumers i and i', $V^i(y_S) = V^{i'}(y_S)$ for every (y_S, x^i_S) in X^i_S, $(y_S, x^{i'}_S)$ in $X^{i'}_S$ and partition S in Z_J.

2.3 Feasibility and Efficiency

An *allocation* is the $J{+}LI$-tuple $(y_S, (x^i_S))$. An allocation $(y_S, (x^i_S))$ is *feasible* if $y_S \in Y_S$, $(y_S, x^i_S) \in X^i_S$ for all consumers i and $C(y_S) + \sum x^i_S \leq w_S$.

A feasible allocation $(y_S, (x^i_S))$, $S \in Z_J$, is *J-Pareto optimal* if there is no other feasible state $(y'_{S'}, (\hat{x}^i_{\hat{S}}))$, $S' \in Z_J$, such that $\hat{z}^i_{\hat{S}} \succeq^i z^i_S$ for all consumers i and $\hat{z}^i_{\hat{S}} \succ^i z^i_S$ for at least one consumer i.

2.4 Equilibrium

In the definition of valuation equilibrium we use the normalized price simplex for the private commodities,

$$\triangle \equiv \{q \in \Re^L_{++} | \sum_{\mathcal{L}} q_l = 1\}.$$

The prices of private commodities are not personalized so the prices of private commodities are represented by the vector valued function $p: Y_J \rightarrow \triangle$.

Definition 2.1 A feasible allocation $(y^\star_{S^\star}, (x^{\star i}_{S^\star}))$ is a *complete personalized valuation equilibrium* on Z_J if there exists a price system $p\colon Y_J \to \triangle$ and a system of complete personalized valuations for consumers $V\colon \mathcal{I} \times Y_J \to \Re$, such that

(1) $\sum V^i(y^\star_{S^\star}) = p_{S^\star}(y^\star_{S^\star})C(y^\star_{S^\star})$, *(budget neutrality)*

(2) for every consumer i, $z^{\star i}_{S^\star}$ maximizes $\succ^i$ on the budget set

$$\{(y_S, x_S) \in X^i_J | p_S(y_S)x_S + V^i(y_S) \le p_S(y_S)w^i\},$$

(3) $y^\star_{S^\star}$ maximizes surplus $\sum V^i(y_S) - p_S(y_S)C(y_S)$ on Y_J.

A *personalized valuation equilibrium* and a *non-personalized valuation equilibrium* are defined analogously.

Definition 2.2 A feasible allocation $(y^\star_{S^\star}, (x^{\star i}_{S^\star}))$ is a *complete personalized cost share equilibrium* if it is a valuation equilibrium and for all consumers i and y in Y, $V^i(y) \ge 0$.

A *personalized cost share equilibrium* and a *non-personalized cost share equilibrium* are defined analogously.

Remark The model of Mas-Colell (1980) can be obtained by letting $J = 1$ and $L = 1$. In this sense the model here is a generalization of the model of Mas-Colell.

However, the model in this paper may also be viewed as a specialization of the model of Mas-Colell. In Mas-Colell the commodity space was any abstract set Y. Consumers were, subject to their budget constraint, free to demand any allocation in Y. Note that an example of an abstract space Y is $\Pi^\star_{Z_J} Y_S$. Restricting attention to abstract spaces with the special form $\Pi^\star_{Z_J} Y_S$ allows the very general form of valuations allowed for in Mas-Colell to be specialized so that the economic content of the model can be expanded.

3 Welfare and Existence

The set of allocations strictly preferred by consumer i to any allocation $z^\star_{S^\star} = (y^\star_{S^\star}, (x^{\star i}_{S^\star}))$ relative to the partition S is

$$P^i_S(z^\star_{S^\star}) = \{(y_S, x^i_S) \in X^i_S | (y_S, x^i_S) \succ^i (y^\star_{S^\star}, x^{\star i}_{S^\star})\}.$$

The set of all allocations strictly preferred by consumer i to any allocation $z^{\star}_{S^{\star}} = (y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}}))$ is

$$P^i(z^{\star}_{S^{\star}}) = \Pi^{\star}_{Z_J} P^i_S(z^{\star}_{S^{\star}}).$$

a.1 for every consumer i, $\succeq^i$continuous, complete, reflexive and transitive.

a.2 for every consumer i, $\succeq^i$is convex and monotonic in the private commodity subspace, for each vector of local public projects under each partition.

a.3S for every consumer i, for all vectors of local public projects y_S such that $X^i_S(y_S) \neq \emptyset, P^i_S(y_S) \neq \emptyset$. (*non-satiation*)

a.4 for each consumer i, for all $(y_S, (x^i_S))$ such that (y_S, x^i_S) is in X^i_S for all i and $x^i_S > 0$, $(y_S, x^i_S) \succ^i (y_{S'}, 0)$ for any $(y_{S'}, 0)$ in $X^i_{S'}$ and for any S' in Z_J. (*essentiality*)

Theorem 3.1 *Under (3S)$_{Z_J}$, if the allocation $(y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}}))$ is a complete personalized, personalized or non-personalized valuation equilibrium on Z_J then it is J-Pareto optimal.*

Theorem 3.2 *Under 1,2,(3S)$_{Z_J}$ and 4, if the allocation $(y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}}))$ is J-Pareto optimal then it is a complete personalized valuation equilibrium on Z_J.*

The set of feasible allocations under any partition S is

$$F_S = \{(y_S, x_S) \in Y_S \times \Re^L | C(y_S) + x_S \leq w\}.$$

a.5S for every consumer i, $F_S \subseteq F_{S^{\star}}$ and $P^i_S(z^{\star}_{S^{\star}}) \subseteq P^i_{S^{\star}}(z^{\star}_{S^{\star}})$ (*nestedness*).

Theorem 3.3 *Under 1,2,(3S)$_{Z_J}$,4 and (5S)$_{Z_J}$, if the allocation $(y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}}))$ is J-Pareto optimal then it is a personalized valuation equilibrium on Z_J.*

a.6 for every pair of consumers i and i', $P^i_{S^{\star}}(z^{\star}_{S^{\star}}) = P^{i'}_{S^{\star}}(z^{\star}_{S^{\star}})$ (*similarity*).

Theorem 3.4 *Under 1,2,(3S)$_{Z_J}$,4,(5S)$_{Z_J}$ and 6, if the allocation $(y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}}))$ is J-Pareto optimal then it is a non-personalized valuation equilibrium on Z_J.*

Example (*Existence*) Consider a model with two private goods. Let there be two consumers, $\mathcal{I} = \{1, 2\}$, each consumer i with consumption sets $X^i_{S_1} = \mathcal{R}^2_+$, $X^i_{S_2} = \mathcal{R}^2_+$ defined relative to the partitions $S_1 = \{\{1, 2\}, \phi\}$ and $S_2 = \{\{1\}, \{2\}\}$. Each consumer has an endowment $w^1 = (3, 0)$ and $w^2 = (0, 3)$.

Let there be only one local public project provided in any of two regions in which any consumer resides, whether the two consumers reside together or apart. Since there is only one local public project provided under each partition, each partition itself can be viewed as a public project, in which case the cost of providing each partition is $C_{S_1} = (1,1)$ and $C_{S_2} = (0,2)$, respectively.

Each consumer has a preference ordering represented by $U^i\left(x^i_{S_1}\right) = 3\sqrt{x^{i1}_{S_1}} + \sqrt{x^{i2}_{S_1}}$ and $U^i\left(x^i_{S_2}\right) = 2\sqrt{x^{i1}_{S_2}} + 2\sqrt{x^{i2}_{S_2}}$ for every private commodity bundle, $x^i = (x^{i1}, x^{i2}) \in \mathcal{R}^2_+$, for $i = 1, 2$.

The sets of feasible allocations are $F_{S_1} = \{x_{S_1} \in \Re^2 \mid x_{S_1} \leq (2,2)\}$ and $F_{S_2} = \{x_{S_2} \in \Re^2 \mid x_{S_2} \leq (3,1)\}$. Nestedness is violated by F_{S_1} and F_{S_2}.

Consider the allocation $x^i_{S_1} = (1,1)$, $i = 1, 2$. Since $x^1_{S_1} + x^2_{S_1} + C_{S_1} = (3,3) = w^1 + w^2$. By inspection $x^i_{S_1} = (1,1)$, $i = 1, 2$ is Pareto optimal (but not the only Pareto optimal allocation). The utility enjoyed by both consumers at the Pareto optimal allocation is $U^i(1,1) = 3\sqrt{1} + \sqrt{1} = 4$, $i = 1, 2$. The Pareto preferred sets generated by $x^i_{S_1} = (1,1)$, $i = 1, 2$ are $P^i_{S_1} = \{x^i_{S_1} \in \Re^2_+ \mid 3\sqrt{x^{i1}_{S_1}} + \sqrt{x^{i2}_{S_1}} > 4\}$ and $P^i_{S_2} = \{x^i_{S_2} \in \Re^2_+ \mid 2\sqrt{x^{i1}_{S_2}} + 2\sqrt{x^{i2}_{S_2}} > 4\}$, $i = 1, 2$. Nestedness is violated by $P^i_{S_1}$ and $P^i_{S_2}$.

Inspection reveals that the prices $p_{S_1} = (3/4, 1/4)$ and $p_{S_2} = (1/2, 1/2)$ seperate the feasible and Pareto preferred sets and that $p_{S_1} = (3/4, 1/4)$ will not seperate the production and Pareto preferred sets under the partition S_2.

The value of the endowment of each consumer at the prices under the partition S_1 is $p_{S_1}.w^1 = 9/4$ and $p_{S_1}.w^2 = 3/4$ and the value of the Pareto optimal allocation under the partition S_1 is $p_{S_1}.x^1_{S_1} = 1$ and $p_{S_1}.x^2_{S_1} = 1$. Budget balance is maintained at the Pareto optimal allocation by imposing valuations of ${V^1}_{S_1} = 5/4$ and ${V^2}_{S_1} = -1/4$ on the two consumers.

The value of the endowment of each consumer at the prices under the partition S_2 is $p_{S_2}.w^1 = 3/2$ and $p_{S_2}.w^2 = 3/2$. The cheapest allocation under S_2 and the prices $p_{S_2} = (1/2, 1/2)$ that is at least as good as $x^i_{S_1} = (1,1)$, $i = 1, 2$ is $x^i_{S_2} = (1,1)$, $i = 1, 2$. The value of $x^i_{S_2} = (1,1)$, $i = 1, 2$ under the prices $p_{S_2} = (1/2, 1/2)$ is unity. To ensure that any allocation under S_2 preferred to $x^i_{S_2} = (1,1)$, $i = 1, 2$ by either consumer is too expensive valuations of ${V^1}_{S_2} = 1/2$ and ${V^2}_{S_2} = 1/2$ must be imposed. □

4 The Core

4.1 Defecting Coalitions and the Core

We allow defecting coalitions to form. We adapt the notion of the Core, introduced by Foley (1970), to economies with local public projects. The notion of the Core we introduce incorporates the idea that all defecting coalitions have access to the production technology and can by some means exclude others from enjoying the public projects they produced.

Each defecting coalition forms a central government that is responsible for the production of all local public projects using the endowment of the defecting coalition. Each defecting coalition may form any jurisdiction structure it wishes.

A *defecting coalition* of consumers consists of a subset of $\mathcal{I}$, $\mathcal{C}$ where $\#\mathcal{C} = I_{\mathcal{C}}$. When consumers defect from the grand coalition so that society is composed of two or more coalitions each defecting coalition $\mathcal{C}$ forms a jurisdiction structure with $J_{\mathcal{C}}$ regions. We use the convention $\mathcal{J}_{\mathcal{C}} = \{1_{\mathcal{C}}, \ldots, j_{\mathcal{C}}, \ldots, J_{\mathcal{C}}\}$.

Consumer residence choice when a member of the defecting coalition $\mathcal{C} \subset \mathcal{I}$ is indicated by the partition of $\mathcal{C}$, $S_{\mathcal{C}}$, where $\#S_{\mathcal{C}} = J_{\mathcal{C}}$. The set of all possible partitions of $\mathcal{C}$ is $Z_{\mathcal{C}}$. Here it is accepted that any partition of all consumer in $\mathcal{I}$, S, may be into any number of regions. All the definitions of Section 2 hold here with respect to any partition of all consumers (the grand coalition) or S in Z.

Associated with each region $j_{\mathcal{C}}$ is a non-empty space $Y^j_{S_{\mathcal{C}}}$ of local public projects, where a local public project is $y^j_{S_{\mathcal{C}}}$. It is possible that no local public project is provided in any region, so $0 \in Y^j_{S_{\mathcal{C}}}$ for every region $j_{\mathcal{C}}$. Often we will write the vector of local public projects $(y^j_{S_{\mathcal{C}}})_{\mathcal{J}}$ as $y_{S_{\mathcal{C}}}$.

The set of all possible vectors of local public projects is $Y_{\mathcal{C}}$, where $Y_{\mathcal{C}} = \Pi_{Z_{\mathcal{C}}} Y_{S_{\mathcal{C}}}$. The cost of any vector of local public projects is given by a function $C\colon Y_{\mathcal{C}} \to \Re^L_+$, for every defecting coalition $\mathcal{C}$. The cost function C is assumed to be proper in $Y_{\mathcal{C}}$. A production program of the defecting coalition is a $J_{\mathcal{C}} + L$-tuple $(y_{S_{\mathcal{C}}}, x_{S_{\mathcal{C}}})$.

If any consumer resides in region $j_{\mathcal{C}}$ then that consumers consumption of local public projects is $(0, \ldots, y^j_{S_{\mathcal{C}}}, \ldots, 0)$ or more simply $\underline{y}^j_{S_{\mathcal{C}}}$. The consumption vector of consumer i, a member of coalition $\mathcal{C}$, when residing in region $j_{\mathcal{C}}$, is $z^i_{S_{\mathcal{C}}}$ where $z^i_{S_{\mathcal{C}}} = (\underline{y}^j_{S_{\mathcal{C}}}, x^i_{S_{\mathcal{C}}})$, where $x^i_{S_{\mathcal{C}}}$ is the vector of private commodities consumed and $\underline{y}^j_{S_\mathcal{C}}$ is the consumption of local public projects by consumer i relative to partition S. Aggregate consumption is therefore $z_{S_\mathcal{C}}$ where $z_{S_\mathcal{C}} = (y_{S_\mathcal{C}}, x_{S_\mathcal{C}})$ and $x_{S_\mathcal{C}} = \sum_\mathcal{C} x^i_{S_\mathcal{C}}$.

Preferences for each consumer i are represented by the complete preordering $\succ^i$ over X^i where X^i is now defined for every partition of every coalition (including the grand coalition) that consumer i is a member of. Without loss of generality, let $X^i = \Pi_{\mathcal{C}\subseteq\mathcal{I}}\Pi^\star_{Z_\mathcal{C}} X^i_{S_\mathcal{C}}$.

It is maintained in this section that there are no increasing returns to coalition size (NIRCS). Defecting coalitions do not have access to a technology superior to the grand coalition. Let $\mathcal{D}$ be any subset of $\mathcal{I}, \mathcal{D} \subseteq \mathcal{I}$.

(NIRCS) *Consider any defecting coalition* $\mathcal{C} \subseteq \mathcal{D}$, *their allocation amongst* $J_\mathcal{C}$ *regions and the set of production opportunities defined with respect to the partition of* $\mathcal{C}$, $F_{S_\mathcal{C}}$. *Consider the residual population* $\mathcal{D}\backslash\mathcal{C}$ *and the set of production opportunities defined with respect to any allocation of these consumers amongst* $J_{\mathcal{D}\backslash\mathcal{C}}$ *regions and the set of production opportunities defined with respect to the partition of* $\mathcal{D}\backslash\mathcal{C}$, $F_{S_{\mathcal{D}\backslash\mathcal{C}}}$. *Then, where* $S_\mathcal{D} = S_\mathcal{C} \cup S_{\mathcal{D}\backslash\mathcal{C}}$, $F_{S_\mathcal{C}} \oplus F_{S_{\mathcal{D}\backslash\mathcal{C}}} \subseteq F_{S_\mathcal{D}}$[4].

Intuitively, no coalitions C and $D\backslash C$ can produce an allocation that their union cannot. In particular, no coalitions C and $I\backslash C$ can produce an allocation that the grand coalition cannot.

An *allocation* for the defecting coalition $\mathcal{C}$ is the $J_\mathcal{C}+LI_\mathcal{C}$-tuple $(y_{S_\mathcal{C}}, (x^i_{S_\mathcal{C}})_\mathcal{C})$. An allocation $(y_{S_\mathcal{C}}, (x^i_{S_\mathcal{C}})_\mathcal{C})$ is *C-feasible* if $y_{S_\mathcal{C}} \in Y_{S_\mathcal{C}}$, $x^i_{S_\mathcal{C}} \in X^i_{S_\mathcal{C}}$ for all consumers i and $C(y_{S_\mathcal{C}}) + \sum_\mathcal{C} x^i_{S_\mathcal{C}} \leq \sum_\mathcal{C} w^i$.

A feasible allocation $(y_S, (x^i_S)_\mathcal{I})$ is *Pareto optimal* if there is no other feasible state $(\hat{y}_{\hat{S}_\mathcal{C}}, (\hat{x}^i_{\hat{S}_\mathcal{C}})_\mathcal{C})$ such that $\hat{z}^i_{\hat{S}_\mathcal{C}} \succeq^i z^i_S$ for all consumers i in $\mathcal{I}$ and $\hat{z}^i_{\hat{S}_\mathcal{C}} \succ^i z^i_S$ for at least one consumer i in $\mathcal{I}$.

An allocation $(y_S, (x^i_S)_\mathcal{I})$ is *blocked* by a coalition $\mathcal{C} \neq \emptyset$ if there exists a $\mathcal{C}$-feasible allocation $(y_{S_\mathcal{C}}, (x^i_{S_\mathcal{C}})_\mathcal{C})$ such that $x^i_{S_\mathcal{C}} \succeq^i x^i_S$ for all consumers i in $\mathcal{C}$ and $x^i_{S_\mathcal{C}} \succ^i x^i_S$ for some consumer i in $\mathcal{C}$. An allocation is in the *core* if it cannot be blocked.

Example (The Core) Consider the example of Section 3. The defecting coalitions are $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2\}$. Each defecting coalition has only one possible partition (or one partition worth considering) $S_{\mathcal{C}_1} = \{\{1\}\}$ and $S_{\mathcal{C}_2} =$

[4] $Y \oplus Z = \{(y, z, x^1 + x^2) \mid (y, x^1) \in Y, (z, x^2) \in Z\}$.

$\{\{2\}\}$.

Assume there is only one local public project provided under each partition, each partition itself can be viewed as a public project, in which case the cost of providing each partition is $C_{S_{\mathcal{C}_1}} = (0,0)$ and $C_{S_{\mathcal{C}_2}} = (0,2)$, respectively.

Each consumer i has a consumption set $X^i_{S_{\mathcal{C}_1}} = \mathcal{R}^2_+$, $X^i_{S_{\mathcal{C}_2}} = \mathcal{R}^2_+$. Each consumer has a preference ordering represented by $U^i\left(x^i_{S_{C_i}}\right) = 2\sqrt{x^{i1}_{S_{C_i}}} + 2\sqrt{x^{i2}_{S_{C_i}}}$ for every private commodity bundle, $x^i = (x^{i1}, x^{i2}) \in \mathcal{R}^2_+$, for $i = 1, 2$.

4.2 Results

Theorem 4.1 *Under $(3S)_Z$, if the allocation $(y^\star_{S^\star}, (x^{\star i}_{S^\star})_{\mathcal{I}})$ is a complete personalized, personalized or non-personalized cost share equilibrium (on Z) and is Pareto optimal then it is in the Core.*

Theorem 4.2 *Under 1,2,$(3S)_{Z_\mathcal{C}}$ and 4, if the allocation $(y^\star_{S^\star}, (x^{\star i}_{S^\star})_{\mathcal{I}})$ it is in the Core then it may be supported as a complete personalized cost share equilibrium on Z when $L = 1$.*

Theorem 4.3 *Under 1,2,$(3S)_Z$,4 and $(5S)_Z$, if the allocation $(y^\star_{S^\star}, (x^{\star i}_{S^\star})_{\mathcal{I}})$ is in the Core then it may be supported as a personalized cost share equilibrium on Z when $L = 1$.*

Theorem 4.4 *Under 1,2,$(3S)_Z$,4,$(5S)_Z$ and 6, if the allocation $(y^\star_{S^\star}, (x^{\star i}_{S^\star})_{\mathcal{I}})$ is in the Core then it may be supported as a non-personalized cost share equilibrium on Z when $L = 1$.*

Theorem 4.5 *There exists an economy such that the core is larger than the set of cost share equilibria when $L > 1$.*

Proof Consider that allocation (y_{S_1}, x_{S_1}) with $x_{1S_1} = x_{2S_1} = (1,1)$. Since $x^1_{S_1} + x^2_{S_1} + C(y_{S_1}) = (3,3) = w^1 + w^2$, (y_{S_1}, x_{S_1}) is a feasible allocation. It is immediate that (y_{S_1}, x_{S_1}) is Pareto optimal. In addition, no consumer can unilaterally improve on (y_{S_1}, x_{S_1}). By NIRCS neither agent has the resources to produce y_{S_1} as producing y_{S_1} would require some of both private goods. Any allocation (y_{S_2}, x_{S_2}) that is feasible for consumer 2 to produce unilaterally can yield consumer 2 a utility no higher than 1, less than the 4 attained at (y_{S_1}, x_{S_1}).

The following price system and valuations support the allocation (y_{S_1}, x_{S_1}) as a valuation equilibrium:

Since $V^2(y_{S_1}) = -1/4 < 0$, (y_{S_1}, x_{S_1}) is not a cost share equilibrium.

The price system $p(y_{S_1})$ is unique. If $V^2(y_{S_1}) \geq 0$ it must be the case that $V^2(y_{S_1}) \leq 1$. For any such valuation system, (y_{S_1}, x_{S_1}) is not a valuation equilibrium.

5 Relationship to the Literature

Why should we care about the model of Section 2? One reason we should care about the model of Section 2 is it has as a special case an important class of models: models with thresh-hold production of public goods. Bliss and Nalebuff (1984), Lagunoff (1994) and others have characterized partial information equilibrium and have designed implementation schemes for models where public goods may only be supplied in some fixed quantity or not supplied and may only be supplied if at least some fixed number of agents (maybe one) agree to contribute towards provision. Bliss and Nalebuff offer as examples: opening a window, donating a library or jumping into save a drowning swimmer. In the model of Lagunoff there are a fixed and finite number of consumers distributed among two regions, who each would have a utility of unity from a local public project supplied and zero if it were not supplied in their region. Each consumer is endowed with one unit of one private good. Consumers may consume all of the private good, in which case their utility from consuming the private good would be some number on the interval $[0, 1)$, and contribute nothing to the production of the local public project or contribute all of their endowment of the private good to the production of their local public project. If not enough consumers contribute then the local public project is not supplied the contributions are lost to the contributors.

Lagunoff considers two social choice mechanisms, a voluntary mechanism and a majority voting mechanism, to decide the contribution level in each of the two regions. Consumers are free to migrate between the two regions and an evolutionary procedure is used to make a prediction about the "winning" social choice mechanism, i.e. the social choice mechanism the consumers migrate to.

How might we specialize the model of Section 2 so that such public "projects" can be represented?

Consider the following specialization of the model of Section 2: Let the number of private goods $L = I$ (the private good endowment of each consumer is differentiated), the number of regions $J = 2$ and the partition of all I consumers amongst the two regions be $\{\mathcal{I}_1, \mathcal{I}_2\}$.

It is convenient to index local public projects by the subset of residents who contribute towards production of their local public project. Any region j with a population I_j has $\#\mathcal{P}(\mathcal{I}_j)$ local public projects, where $\mathcal{P}(\mathcal{I}_j)$ is the power set of $\mathcal{I}_j$. Let $\mathcal{D}_j$ be any element of $\mathcal{P}(\mathcal{I}_j)$. n is the "thresh-hold". The set of local public projects available to region j relative to the partition S is $\mathcal{P}(\mathcal{I}_j)$.

The preferences of any consumer i residing in region j are represented by

$$U^i = \begin{cases} 0; & \text{if} \quad i \in \mathcal{D}_j \quad \text{and} \quad \#\mathcal{D}_j < n, \\ \alpha^i, \alpha^i \in [0,1)\,; & \text{if} \quad i \in \mathcal{I}_j \backslash \mathcal{D}_j \quad \text{and} \quad \#\mathcal{D}_j < n, \\ 1; & \text{if} \quad i \in \mathcal{D}_j \quad \text{and} \quad \#\mathcal{D}_j \geq n, \\ 1 + \alpha^i, \alpha^i \in [0,1)\,; & \text{if} \quad i \in \mathcal{I}_j \backslash \mathcal{D}_j \quad \text{and} \quad \#\mathcal{D}_j \geq n. \end{cases}$$

The cost of production is represented by a function $C \colon \mathcal{P}(\mathcal{I}_j) \to \mathcal{P}(\mathcal{I}_j)$ such that

$$C(\mathcal{D}_1, \mathcal{D}_2) = \begin{cases} (\mathcal{D}_1, \mathcal{D}_2) & \text{if } \#\mathcal{D}_1 \geq n \text{ and } \#\mathcal{D}_2 \geq n, \\ (\mathcal{D}_1, \emptyset) & \text{if } \#\mathcal{D}_1 \geq n \text{ and } \#\mathcal{D}_2 < n, \\ (\emptyset, \mathcal{D}_2) & \text{if } \#\mathcal{D}_1 < n \text{ and } \#\mathcal{D}_2 \geq n, \\ (\emptyset, \emptyset) & \text{if } \#\mathcal{D}_1 < n \text{ and } \#\mathcal{D}_2 < n. \end{cases}$$

6 Conclusion

Some effort has already been made in the literature to show that Lindahl pricing schemes may be implemented as lump sum taxes rather than linear prices. For instance, Wooders (1992) has shown that any Lindahl equilibrium, in a sufficiently replicated economy, may be implemented using lump sum taxes. Barro and Romer (1987) call the equivalence between the linear pricing mechanism and the lump sum pricing mechanism the *package deal effect*. By a specialisation of the class of models presented here, this paper shows that the package deal effect holds for finite economies with local public goods. For instance, it is immediate that the package deal effect holds for the class of models in Manning (1993).

7 Appendix

Proof of Theorem 3.1

Suppose that $(y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}})_{\mathcal{I}})$ is a valuation equilibrium but is not Pareto optimal. Suppose that $(y_S, (x^i_S)_{\mathcal{I}})$ is feasible and Pareto dominates $(y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}})_{\mathcal{I}})$. We know that such an allocation may exist by $a.3$. This implies that for at least one i

$$p_S(y_S)x^i_S + V^i(y_S) > p_S(y_S)w^i,$$

and for all i

$$p_S(y_S)x^i_S + V^i(y_S) \geq p_S(y_S)w^i.$$

Therefore

$$p_S(y_S)\sum_{\mathcal{I}} x^i_S + \sum_{\mathcal{I}} V^i(y_S) > p_S(y_S)w,$$

which implies

$$\sum_{\mathcal{I}} V^i(y_S) > p_S(y_S)[\sum_{\mathcal{I}}(w^i - x^i_S)] = p_S(y_S)C(y_S),$$

implying that $y^{\star}_{S^{\star}}$ does not maximize surplus $\sum_{\mathcal{I}} V^i(y_S) - p_S(y_S)C(y_S)$ and so contradicting condition (3) of the definition of valuation equilibrium.□

Proof of Theorem 3.2

Part 1 Let $z^{\star}_{S^{\star}} = (y^{\star}_{S^{\star}}, (x^{\star i}_{S^{\star}})_{\mathcal{I}})$ be a Pareto optimal allocation. Define

$$P^i_S(y_S) \equiv \{x^i_S \in X^i_S(y_S) | (y_S, x^i_S) \succ^i (y^{\star}_{S^{\star}}, x^{\star i}_{S^{\star}})\}$$
$$\text{and } R^i_S(y_S) \equiv \{x^i_S \in X^i_S(y_S) | (y_S, x_S) \succeq^i (y^{\star}_{S^{\star}}, x^{\star i}_{S^{\star}})\}.$$

The following are immediate: $P^i_S(y_S)$ and $R^i_S(y_S)$ are nonempty, open (closed), convex and bounded below. Define

$$P_S(y_S) \equiv \sum_{\mathcal{I}} P^i_S(y_S) + C(y_S) - w \text{ and } R_S(y_S) \equiv \sum_{\mathcal{I}} R^i_S(y_S) + C(y_S) - w.$$

The following are immediate: $P_S(y_S)$ and $R_S(y_S)$ are nonempty, open (closed), convex and bounded below. In addition $R^i_S(y_S)$ is the closure of $P^i_S(y_S)$ and $R_S(y_S)$ is the closure of $P_S(y_S)$.

Claim *Under 1, 2 and $(3S)_Z$ for any partition S in Z and any local public project y_S in Y_S there exists a price vector $p'_S(y_S)$ in $\triangle$ and a vector of private goods $x'^i_S(y_S)$ in $R^i_S(y_S)$ such that*

(i) $p'_S(y_S)x'^i_S(y_S) = \inf\{p_S(y_S)x_S | x_S \in P^i_S(y_S)\}$, *for all* i,

(ii) $\sum_{\mathcal{I}} x'^i_S(y_S) + C(y_S) - w \geq 0$,

(iii) $x'^i_S(y_S) = x^{\star i}_{S^\star}(y^\star_{S^\star}) = x^{\star i}_{S^\star}$, *for all consumers* i, *if* $y_S = y^\star_{S^\star}$.

(See Diamantaras and Gilles (1994) for a proof).

Part 2 We define a valuation function $V\colon \mathcal{I} \times Y \to \Re$ by

$$V^i(y_S) = \begin{cases} p_S(y_S)w^i - p_S(y_S)x'^i_S(y_S); & \text{for all } y_S \text{ such that } X^i_S(y_S) \neq \emptyset, \text{for all } i, \\ 0; & \text{otherwise,} \end{cases}$$

where

$$p_S(y_S) = \begin{cases} p'_S(y_S); & \text{for all } y_S \text{ such that } X^i_S(y_S) \neq \emptyset, \text{for all } i, \\ 0; & \text{otherwise.} \end{cases}$$

$V^i(y_S)$ is finite by a.3. We now check the three requirements of Definition 2.1.

Condition (1) By the feasibility of $(y^\star_{S^\star}, (x^{\star i}_{S^\star})_{\mathcal{I}})$ and the definition of V,

$$\sum_{\mathcal{I}} V^i(y^\star_{S^\star}) = p'_{S^\star}(y^\star_{S^\star}) \sum_{\mathcal{I}} w^i - p'_{S^\star}(y^\star_{S^\star}) \sum_{\mathcal{I}} x^{\star i}_{S^\star} = p'_{S^\star}(y^\star_{S^\star})C(y^\star_{S^\star}).$$

Condition (3) By claim (ii) and $p'_S(y_S) \gg 0$, for $y_S \neq y^\star_{S^\star}$ and y_S such that $X^i_S(y_S) \neq \emptyset$, for all i,

$$p'_S(y_S) \sum_{\mathcal{I}} x'^i_S(y_S) + p'_S(y_S)C(y_S) \geq p'_S(y_S)w,$$

which implies

$$\sum_{\mathcal{I}} V^i(y_S) = p'_S(y_S)w - p'_S(y_S) \sum_{\mathcal{I}} x'^i_S(y_S) \leq p'_S(y_S)C(y_S),$$

while

$$\sum_{\mathcal{I}} V^i(y^\star_{S^\star}) = p'_{S^\star}(y^\star_{S^\star})w - p'_{S^\star}(y^\star_{S^\star}) \sum_{\mathcal{I}} x'^i_{S^\star}(y^\star_{S^\star}) \leq p'_{S^\star}(y^\star_{S^\star})C(y^\star_{S^\star}).$$

Therefore

$$0=\sum_{\mathcal{I}} V^i(y^\star_{S^\star}) - p'_{S^\star}(y^\star_{S^\star})C(y^\star_{S^\star}) \geq \sum_{\mathcal{I}} V^i(y_S) - p'_S(y_S)C(y_S).$$

If y_S is such that $X^i_S(y_S)=\emptyset$ for some i then $V^i(y_S)=0$ for all i, by assumption. Therefore

$$0=\sum_{\mathcal{I}} V^i(y^\star_{S^\star}) - p_{S^\star}(y^\star_{S^\star})C(y^\star_{S^\star}) \geq \sum_{\mathcal{I}} V^i(y_S) - p_S(y_S)C(y_S)$$

as $\sum_{\mathcal{I}} V^i(y_S) - p_S(y_S)C(y_S) = -p_S(y_S)C(y_S) \leq 0$.

It is immediate that condition (3) of the definition of valuation equilibrium holds.

Condition (2) Note that since $V^i_{S^\star}(y^\star_{S^\star}) = p'_{S^\star}(y^\star_{S^\star})[w^i - x^{\star i}_{S^\star}]$ it immediately follows that $p_{S^\star}(y^\star_{S^\star})x^{\star i}_{S^\star} + V^i(y^\star_{S^\star}) = p_{S^\star}(y^\star_{S^\star})w^i$ for every consumer i. Thus $(y^\star_{S^\star}, x^{\star i}_{S^\star})$ is indeed in the budget set of each consumer i.

Consider any (y_S, x^i_S) in $P^i_S(z^{\star i}_{S^\star})$, for any consumer i. By the claim above, associated with the public project y_S are vectors $x'^i_S(y_S)$ and $p_S(y_S)$. By definition of $x'^i_S(y_S)$ and $p'_S(y_S)$, $p'_S(y_S)x^i_S > p'_S(y_S)x'^i_S(y_S)$, which implies

$$p'_S(y_S)x^i_S + V^i(y_S) = p'_S(y_S)x^i_S + p'_S(y_S)w^i - p'_S(y_S)x'^i_S(y_S) > p'_S(y_S)w^i > 0,$$
$$\text{by } p'_S(y_S) \gg 0.$$

Therefore (y_S, x^i_S) is not in the budget set of consumer i.□

Proof of Theorem 3.3

Part 1 As in proof of Theorem 3.2, Part 1.

Part 2 We define the valuation function $V\colon \mathcal{I}\times Y \to \Re$ by

$$V^i(y_S)=\begin{cases} p_{S^*}(y_{S^*})w^i - p_{S^*}(y_{S^*})x'^i_{S^*}(y_{S^*}); \\ \qquad\qquad \text{for all } y_{S^*} \text{such that } X^i_{S^*}(y_{S^*})\neq\emptyset, \text{for all } i, \\ V^i(y_{S^*}); \\ \text{where } y_S = y_{S^*} \text{and is such that } X^i(y_S)\neq\emptyset, \text{for all } i, \\ 0;\ \text{otherwise.} \end{cases}$$

where

$$p_S(y_S)=\begin{cases} p'_S(y_S); & \text{for all } y_S \text{ such that } X^i_{S^*}(y_{S^*})\neq\emptyset, \text{for all } i, \\ p'_{S^*}(y_{S^*}); & \text{where } y_S = y_{S^*} \text{and is such that } X^i(y_S)\neq\emptyset, \text{for all } i, \\ 0; & \text{otherwise.} \end{cases}$$

Condition (1) By the feasibility of $(y^\star_{S^*}, (x^{\star i}_{S^*})_{\mathcal{I}})$ and the definition of V,

$$\sum_{\mathcal{I}} V^i(y^\star_{S^*}) = p'_{S^*}(y^\star_{S^*})\sum_{\mathcal{I}} w^i - p'_{S^*}(y^\star_{S^*})\sum_{\mathcal{I}} x^{\star i}_{S^*} = p'_{S^*}(y^\star_{S^*})C(y^\star_{S^*}).$$

Condition (3) By claim (ii) and $p_{S^*}(y_{S^*}) \gg 0$, for $y_{S^*} \neq y^\star_{S^*}$ and y_{S^*} such that $X^i_{S^*}(y_{S^*}) \neq \emptyset$, for all i,

$$p'_{S^*}(y_{S^*})\sum_{\mathcal{I}} x'^i_{S^*}(y_{S^*}) + p'_{S^*}(y_{S^*})C(y_{S^*}) \geq p'_{S^*}(y_{S^*})w,$$

which implies

$$\sum_{\mathcal{I}} V^i(y_{S^*}) = p'_{S^*}(y_{S^*})w - p'_{S^*}(y_{S^*})\sum_{\mathcal{I}} x'^i_{S^*}(y_{S^*}) \leq p'_{S^*}(y_{S^*})C(y_{S^*}),$$

while

$$\sum_{\mathcal{I}} V^i(y^\star_{S^*}) = p'_{S^*}(y^\star_{S^*})w - p'_{S^*}(y^\star_{S^*})\sum_{\mathcal{I}} x^{\star i}_{S^*}(y^\star_{S^*}) = p'_{S^*}(y^\star_{S^*})C(y^\star_{S^*}).$$

Therefore

$$0 = \sum_{\mathcal{I}} V^i(y^\star_{S^*}) - p'_{S^*}(y^\star_{S^*})C(y^\star_{S^*}) \geq \sum_{\mathcal{I}} V^i(y_{S^*}) - p'_{S^*}(y_{S^*})C(y_{S^*}).$$

For any $y_S = y_{S^*}$ such that $X^i_S(y_S) \neq \emptyset$, for all i, since since $F_S \subseteq F_{S^*}$ for all S,

$$0 = \sum_{\mathcal{I}} V^i(y^\star_{S^*}) - p'_{S^*}(y^\star_{S^*})C(y^\star_{S^*}) \geq \sum_{\mathcal{I}} V^i(y_{S^*}) - p'_{S^*}(y_{S^*})C(y_{S^*}) \geq \sum_{\mathcal{I}} V^i(y_S) - p'_S(y_S)C(y_S)$$

for any y_S in Y_S as $p'_S(y_S) = p'_{S^*}(y_{S^*}) \gg 0$, $V^i(y_S) = V^i(y_{S^*})$ and $C(y_S) \geq C(y_{S^*})$.

If y_S is such that $X^i_S(y_S) = \emptyset$ for some i then $V^i(y_S) = 0$ for all i, by assumption. Therefore

$$0 = \sum_{\mathcal{I}} V^i(y^\star_{S^*}) - p_{S^*}(y^\star_{S^*})C(y^\star_{S^*}) \geq \sum_{\mathcal{I}} V^i(y_S) - p_S(y_S)C(y_S)$$

as $\sum_{\mathcal{I}} V^i(y_S) - p_S(y_S)C(y_S) = -p_S(y_S)C(y_S) \leq 0$.

It is immediate that condition (3) of the definition of valuation equilibrium holds.

Condition (2) By the argument in Theorem 3.2, $(y^\star_{S^*}, x^{\star i}_{S^*})$ is indeed in the budget set of each consumer i.

Consider any $(y_{S^*}, x^i_{S^*})$ in $P^i_{S^*}(z^{\star i}_{S^*})$, for any consumer i. By the claim above (in Theorem 3.2), associated with the public project y_{S^*} are vectors $x'^i_{S^*}(y_{S^*})$ and $p_{S^*}(y_{S^*})$. By definition of $x'^i_{S^*}(y_{S^*})$ and $p'_{S^*}(y_{S^*})$, $p'_{S^*}(y_{S^*})x^i_{S^*} > p'_{S^*}(y_{S^*})x'^i_{S^*}(y_{S^*})$, which implies

$$\begin{aligned} p'_{S^*}(y_{S^*})x^i_{S^*} + V^i(y_{S^*}) &= p'_{S^*}(y_{S^*})x^i_{S^*} + p'_{S^*}(y_{S^*})w^i - p'_{S^*}(y_{S^*})x'^i_{S^*}(y_{S^*}) \\ &> p'_{S^*}(y_{S^*})w^i > 0, \text{by } p'_{S^*}(y_{S^*}) \gg 0. \end{aligned}$$

Therefore $(y_{S^*}, x^i_{S^*})$ is not in the budget set of consumer i.

Consider any (y_S, x^i_S) in $P^i_S(z^{\star i}_{S^*})$. Since $P^i_S(z^{\star i}_{S^*}) \subseteq P^i_{S^*}(z^{\star i}_{S^*})$, $(y_{S^*}, x^i_{S^*}) \in P^i_{S^*}(z^{\star i}_{S^*})$ where $y_{S^*} = y_S$ and $x^i_{S^*} = x^i_S$. Since, by the argument above, $(y_{S^*}, x^i_{S^*})$ lies outside the budget set so would (y_S, x^i_S) by construction of V^i.□

Proof of Theorem 3.4

Part 1 As in the proof of Theorem 3.2, Part 1.

Part 2 We define the valuation function $V : \mathcal{I} \times Y \to \Re$ by

$$V^i(y_S) = \begin{cases} p_{S^*}(y_{S^*})w^i - p_{S^*}(y_{S^*})x'^i_{S^*}(y_{S^*}); \\ \qquad \text{for all } y_{S^*} \text{such that } X^i_{S^*}(y_{S^*}) \neq \emptyset, \text{for all } i, \\ V^i(y_{S^*}); \\ \text{where } y_S = y_{S^*} \text{and is such that } X^i(y_S) \neq \emptyset, \text{for all } i, \\ \qquad 0; \text{ otherwise.} \end{cases}$$

where

$$p_S(y_S) = \begin{cases} p'_S(y_S); & \text{for all } y_S \text{such that } X^i_S(y_S) \neq \emptyset, \text{for all } i, \\ p'_{S^*}(y_{S^*}); & \text{where } y_S = y_{S^*} \text{and is such that } X^i(y_S) \neq \emptyset, \text{for all } i, \\ 0; & \text{otherwise.} \end{cases}$$

Conditions 1,2 and 3 follow by Theorem 3.2. In addition, by $P^i_{S^*}(z^{\star i}_{S^*}) = P^{i'}_{S^*}(z^{\star i'}_{S^*})$, $x'^i_{S^*}(y_{S^*}) = x'^{i'}_{S^*}(y_{S^*})$, and so $V^i(y_S) = V^{i'}(y_S)$ for all pairs of consumers i and i' and $(y_S, x^i_S) \in X^i_S$, $(y_S, x^{i'}_S) \in X^{i'}_S$.□

Proof of Theorem 4.1

Suppose that $(y^\star_{S^*}, (x^{\star i}_{S^*})_{\mathcal{I}})$ is an extended cost share equilibrium but not in the core. This implies that there exists a defecting coalition $\mathcal{C} \subset \mathcal{I}$

that can provide a $\mathcal{C}$-feasible allocation $(y_{S_\mathcal{C}}, (x^i_{S_\mathcal{C}})_\mathcal{C})$ that $\mathcal{C}$-Pareto dominates $(y^\star_{S^\star}, (x^{\star i}_{S^\star})_\mathcal{I})$. We know that such an allocation may exist by a.3. This implies for at least one i in $\mathcal{C}$

$$p_{S_\mathcal{C}}(y_{S_\mathcal{C}})x^i_{S_\mathcal{C}} + V^i(y_{S_\mathcal{C}}) > p_{S_\mathcal{C}}(y_{S_\mathcal{C}})w^i,$$

and for all i

$$p_{S_\mathcal{C}}(y_{S_\mathcal{C}})x^i_{S_\mathcal{C}} + V^i(y_{S_\mathcal{C}}) \geq p_{S_\mathcal{C}}(y_{S_\mathcal{C}})w^i.$$

Therefore

$$p_{S_\mathcal{C}}(y_{S_\mathcal{C}})\sum_\mathcal{C} x^i_{S_\mathcal{C}} + \sum_\mathcal{C} V^i(y_{S_\mathcal{C}}) > p_{S_\mathcal{C}}(y_{S_\mathcal{C}})\sum_\mathcal{C} w^i,$$

which implies

$$\sum_\mathcal{C} V^i(y_{S_\mathcal{C}}) > p_{S_\mathcal{C}}(y_{S_\mathcal{C}})[\sum_\mathcal{C}(w^i - x^i_{S_\mathcal{C}})] = p_{S_\mathcal{C}}(y_{S_\mathcal{C}})C(y_{S_\mathcal{C}}),$$

implying that $y^\star_{S^\star}$ does not maximize surplus $\sum_\mathcal{I} V^i(y_S) - p_S(y_S)C(y_S)$ and so contradicting condition (3) of the definition of an extended cost share equilibrium. □

Proof of Theorem 4.2

We define a valuation function $V\colon \mathcal{I} \times Y \to \Re$ by

$$V^i(y_S) = \begin{cases} w^i - x'^i_S(y_S); & \text{for all } y_S \text{ such that } X^i_S(y_S) \neq \emptyset, \text{for all } i, \\ 0; & \text{otherwise.} \end{cases}$$

$V^i(y_S)$ is finite by a.3. Conditions 1, 2 and 3 are immediate by the proof to Theorem 3.2. □

Proof of Theorem 4.3

Immediate. □

Proof of Theorem 4.4

Immediate. □

References

1. Barro, R.J. and P.M. Romer, "Ski-Lift Pricing, with Applications to Labour and other Markets," *American Economic Review* 77 (1987), 875-890.

2. Bewley, T., "A Critique of Tiebout's Theory of Local Public Expenditures," *Econometrica* 49 (1981), 713-740.

3. Bliss and B. Nalebuff, "Dragon Slaying and Ballroom Dancing," *Journal of Public Economics*, 1984

4. Diamantaras, D. and R.P. Gilles, "Efficiency in Economies with a Public Project", Department of Economics, Temple University, Philadelphia, PA 19122 and Department of Economics, VPI and SU, Blacksburg, VA 24061, 1992.

5. Diamantaras, D. and R.P. Gilles, "The Pure Theory of Public Goods: Efficiency, Decentralization and the Core," Working Paper E94-01, Virginia Polytechnic Institute and State University, Blacksburgh, VA, 1994

6. Diamantaras, D., R.P. Gilles and S. Scotchmer, "Decentralization of Pareto Optima in Economies with Public Projects and Nonessential Private Goods," Discussion Paper 9453, CentER for Economic Research, Tilburg, Netherlands, revised 1995

7. Diamantaras, D. and R.P. Gilles, "The Pure Theory of Public Goods: Core Equivalence," Working Paper E94-10, Department of Economics, VPI and SU, Blacksburg, VA 24061, 1994.

8. Ellickson, B., "A Generalization of the Pure Theory of Public Goods," *American Economic Review* 63 (1973), 417-432.

9. Gilles, R.P. and S. Scotchmer, "Decentralization in Replicated Club Economies with Multiple Private Goods," Working Paper E95-05, Department of Economics, VPI&SU, Blacksburg.

10. Gilles, R.P. and S. Scotchmer, "Decentralization in Club Economies with Non-Samuelsonian Local Public Goods," Working Paper E95-04, Department of Economics, VPI&SU, Blacksburg.

11. Lagunoff, R., “On the Dynamic Selection of Mechanisms for Provision of Public Projects,” typescript, 1994.

12. Manning, J.R.A., “Local Public Goods: A Theory of the First Best”, P.H.D. thesis, University of Rochester, 1993.

13. Manning, J.R.A., “Efficiency in Economies with Jurisdictions and Public Projects”, typescript, 1993.

14. Manning, J.R.A., “Local Public Goods: First Best Allocations and Supporting Prices”, typescript, 1994.

15. Manning, J.R.A., “A General Model of Valuation and Pure Public Projects”, typesript, 1995.

16. Mas-Colell, A., “Efficiency and Decentralization in the Pure Theory of Public Goods”, *Quarterly Journal of Economics* 94 (1980), 625-641.

17. Tiebout, C., “A Pure Theory of Local Public Expenditures,” *Journal of Political Economy* 64 (1956), 416-424.

18. Wooders, M.H., “Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good,” *Journal of Economic Theory* 18 (1978), 328-348.

19. Wooders, M.H., “The Tiebout Hypothesis: Near Optimality in Local Public Good Economies,” *Econometrica* 48 (1980), 1467-1485.

20. Wooders, M.H., “A Tiebout Theorem,” *Mathematical Social Sciences* 18 (1989), 33-55.

21. Wooders, M.H., “On Convergence of Cores to Lindahl Equilibrium Outcomes,” Department of Economics, University of Toronto, 1992.